A STRUCTURE THEOREM FOR BOUNDARY-TRANSITIVE GRAPHS WITH INFINITELY MANY ENDS

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ABSTRACT

We prove a structure theorem for locally finite connected graphs X with infinitely many ends admitting a non-compact group of automorphisms which is transitive in its action on the space of ends, Ω_X. For such a graph X, there is a uniquely determined biregular tree T (with both valencies finite), a continuous representation $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(T)$ with compact kernel, an equivariant homeomorphism $\lambda : \Omega_X \to \Omega_T$, and an equivariant map $\tau : \mathrm{Vert}(X) \to \mathrm{Vert}(T)$ with finite fibers. Boundary-transitive trees are described, and some methods of constructing boundary-transitive graphs are discussed, as well as some examples.

Introduction

Experience shows that it is usually difficult to analyze the simple random walk on a graph X and the spectral theory of its associated averaging operator P_X acting in $\ell^2(X)$ without some symmetry assumptions on the graph. In the case of graphs arising as Bruhat – Tits buildings of p-adic Lie groups of split rank one, namely biregular trees, the group of automorphisms of X acts transitively on the boundary of X (the space of ends) and this fact plays a crucial role in solving both problems. Locally finite graphs with infinitely many ends are a natural generalization of trees, and it is natural to adopt the symmetry assumptions that the group $G = \mathrm{Aut}(X)$ of graph automorphisms of X is non-compact (in the compact-open topology), and in view of the previous remark, that G acts transitively on the space of ends Ω_X. Such graphs we call boundary-transitive

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graphs. It is natural to call their automorphism groups totally disconnected topological groups of rank one, and we will consider these groups, the harmonic analysis and the random walk on the graph elsewhere. Here we consider the problems of describing the structure of such graphs and constructing examples. As to the first problem, we prove the following structure theorem: given X, there exists a uniquely determined bi-regular tree T, with both valencies finite, a continuous representation $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(T)$, a homeomorphism $\lambda : \Omega_X \to \Omega_T$ which is equivariant with respect to φ, and a map $\tau : V(X) \to V(T)$ from the vertices of X to the vertices of T which is again equivariant with respect to φ, and has finite fibers. $\varphi(G)$ has either one or two orbits of vertices in T, and therefore by equivariance the set of fibers falls into either one or two $\varphi(G)$-orbits. The maps τ and λ are compatible in the sense that $x_n \in X$ and $x_n \to \omega \in \Omega_X$ implies $\tau(x_n) \to \lambda(\omega)$. Concerning the second problem, we describe several ways in which boundary-transitive graphs can be modified to yield other boundary-transitive graphs, for example, appropriately appending a finite graph, adding edges connecting vertices at a fixed distance, taking the dual of the graph, and also describe some examples arising naturally as the 1-skeleton of universal covers of 2-dimensional simplicial complexes with a finitely generated free fundamental group. Also, we show that boundary-transitive trees are subdivisions of biregular trees, and that a free product of two finite graphs is boundary-transitive iff both graphs are complete on their vertices.

The construction of a tree T and a representation $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(T)$ is a fundamental result in the theory of graphs with infinitely many ends and is due to M. J. Dunwoody (see [2]). We show that φ is continuous and that in our case T is locally finite. In the construction of the boundary map $\lambda : \Omega_X \to \Omega_T$ we use some arguments of Woess (see [5]).

1. Let $X = (V, E)$ be a locally finite graph, with vertex set V and a set of unoriented edges E, without loops or multiple edges. Denote by δA the coboundary of a set of vertices $A \subset X$, namely the set of edges having one vertex in A and one in $X \setminus A \overset{d}{=} A^*$. Assume there exists a set of vertices D, with D and D^* both infinite, such that $\delta D = \delta D^*$ is finite, in which case D will be called a cut. If $|\delta D|$(= the number of elements in δD) is as small as possible subject only to D and D^* being infinite, D will be called a narrow cut.

The space of ends Ω of X is constructed as follows: given a finite subgraph F of X, $X \setminus F$ decomposes to finitely many connected components, by the local finiteness of X, and denote the set of components by C_F. The set $\{C_F | F \text{ a finite subgraph of } X\}$ is directed w.r.t. set inclusion, and $\Omega_X = \Omega$ is defined as its inverse limit. By construction, Ω is a totally disconnected compact metrisable space, and $X \cup \Omega$ becomes a compact totally disconnected metrisable space in which X is discrete open and dense and $\partial X = \Omega$. If we define two infinite paths in X to be equivalent iff for every finite subgraph F, almost all (that is, all but finitely many) of their vertices belong to the same component of C_F then there is a natural identification of Ω and the set of equivalence classes.

We can now appeal to the following:

THEOREM 1 ([2]): *If X has a cut, then X has a narrow cut D with the property that for every graph-automorphism $g \in G \stackrel{d}{=} \mathrm{Aut}(X)$ at least one of the following holds:*

$$D \subset gD, \quad D \subset gD^*, \quad D^* \subset gD, \quad D^* \subset gD^*.$$

The existence of such a cut has been utilised by Dunwoody to obtain an explicit construction of a tree T and an explicit representation $\varphi : G \to \mathrm{Aut}(T)$. We recall the details of the construction [1,2]: Define $\mathcal{E} = \{gD, gD^* | g \in G\} =$ the set of G-translates of the narrow cut D and its complement D^*. $\mathcal{E}$ has an involution $\mathcal{E} \ni A \mapsto A^* \in \mathcal{E}$, and set inclusion $\subseteq$ induces a partial order on $\mathcal{E}$, such that the system $(\mathcal{E}, \subseteq, *)$ satisfies the following five properties of a partial order $\leq$ with an involution $A \mapsto \bar{A}$ on a set $\mathcal{A}$:

(1) $A \leq B \Rightarrow \bar{B} \leq \bar{A}$;

(2) for no pair A, B do both $A \leq B$ and $\bar{A} \leq B$ hold;

(3) for no pair A, B do both $A \leq B$ and $A \leq \bar{B}$ hold;

(4) for any pair A, B, at least one of the following holds:

$$A \leq B, \quad A \leq \bar{B}, \quad \bar{A} \leq B, \quad \bar{A} \leq \bar{B};$$

(5) if $A \leq B$, then there are only finitely many C's s.t. $A \leq C \leq B$.

Obviously, $(\mathcal{E}, \subseteq, *)$ satisfies the first three properties, and it satisfies the fourth by Dunwoody's Theorem 1. The fifth property follows from the following:

THEOREM 2 ([4]): *If $\{C_i | i \geq 0\}$ is a descending sequence of narrow cuts, and $\bigcap_{i \geq 0} C_i \neq \phi$ then there exists $n \geq 0$ s.t. $C_n = C_{n+j} \quad \forall j \geq 0$.*

Given such a system define $A < B$ iff $A \leq B$ and $A \neq B$, and $A \prec B$ iff $A < B$ and there is no C s.t. $A < C < B$. The relation: $A \sim B \Leftrightarrow A = B$ or $A \prec \bar{B}$, is an equivalence relation [1], and denoting by $[A]$ the equivalence class of A, take $V = \mathcal{A}/\sim = \{[A] | A \in \mathcal{A}\}$ to be a set of vertices of an oriented graph T, with edge set $\mathcal{A}$, by defining the source and target functions s and t: $\mathcal{A} \to \mathcal{A}/\sim = V$

of edges $A \in \mathcal{A}$ with the formulas: $s(A) = [\bar{A}]$ $t(A) = [A]$. $(V, \mathcal{A}, s, t)$ is an oriented tree: $A \mapsto (s(A), t(A))$ is injective since $A \sim B$ and $\bar{A} \sim \bar{B}$ implies either $A = B$ or $A \prec \bar{B} \prec A$, the graph is connected by property (5) and has no cycles since $\leq$ is a partial order. Moreover, two edges A, B in the tree can be joined by an oriented path iff $A \leq B$. More concretely, for the system $(\mathcal{E}, \subseteq, *)$ obtained from X, the tree can be described as follows: the relation $A \prec B$ means that $A \underset{\neq}{\subset} B$ but there is no other $C \in \mathcal{E}$ s.t. $A \underset{\neq}{\subset} C \underset{\neq}{\subset} B$, in which case we will call A a subset of index one in B, and the equivalence relation $A \sim B$ means: either $A = B$ or A is a subset of index one in $B^* = X \setminus B$. Therefore, the edges of T (sets $A \in \mathcal{E}$) whose target vertex is the source vertex of $D \in \mathcal{E}$ are precisely the edges given by subsets A_i of index one in D, and the edges of T whose target vertex is the target vertex of D are precisely the edges given by subsets B_j of index one in D^*, see Fig. 1.

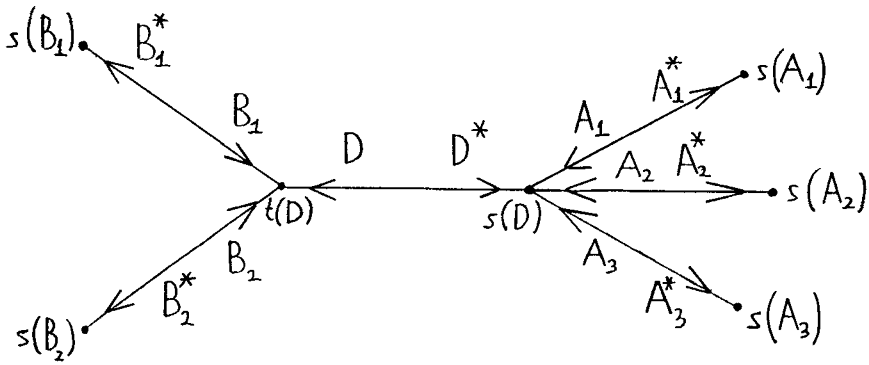

Fig. 1

In this tree of inclusions, a directed path $C_1C_2C_3 \cdots C_n$ going from $s(C_1)$ to $t(C_n)$ according to the arrows corresponds to a chain of index-one (increasing) inclusions $C_1 < C_2 < \cdots < C_n$, and an antidirected path $C_1C_2C_3 \cdots C_n$ going from $t(C_1)$ to $s(C_n)$ against the arrows corresponds to the chain of index-one (decreasing) inclusions $C_1 > C_2 > \cdots > C_n$. We note that since $G = \mathrm{Aut}(X)$ preserves the partial ordering $\subseteq$ on $\mathcal{E}$ and commutes with the involution $*$, the equivalence relation $\sim$ is G-invariant, and so there is a natural action of G on T by graph-automorphisms, and moreover, this action in T has either one or two orbits of edges V depending on whether G acts with or without inversions, which is determined by whether or not G has an involution σ satisfying $\sigma(D) =$

D^*. It is clear that the G action has either one or two orbits of vertices in T, again depending on the existence of an involution σ as above. Denote the representation $G \to \mathrm{Aut}(T)$ by φ.

PROPOSITION 1: *φ is a continuous representation.*

Proof: We take the compact - open topology on $\mathrm{Aut}(T)$, and clearly any increasing sequence $A_n \subset T$ of finite sets with $\bigcup_n A_n = T$ defines a base $H_n \leq \mathrm{Aut}(T)$ of neighbourhoods of the identity, where $H_n = \{g \in \mathrm{Aut}(T) | ga = a \quad \forall a \in A_n\}$ since T is discrete in its natural metric.

Recall now the following result which follows easily from Theorems 1 and 2.

LEMMA 3 ([1]): *The stabiliser of a narrow cut $D \in \mathcal{E}$ contains the stabiliser of any edge $e \in \delta D$.*

It follows that φ is continuous, since if $g_n \to I$ in $\mathrm{Aut}(X)$, g_n fixes larger and larger finite sets of vertices and edges in X, so g_n fixes the narrow cuts which contain these edges in their boundary, and so $\varphi(g_n)$ fixes larger and larger finite sets of edges in T, so $\varphi(g_n) \to I$. □

Fixing a reference vertex $0 \in T$, consider the space $\Lambda_T =$ the set of all infinite geodesics starting at 0. There is an associated metric $d_0(\omega, \omega')$ on Λ_T, which assigns the value e^{-n} where n is the distance in T between 0 and the point from which the unique geodesics from 0 to ω and ω' diverge. Clearly, $\mathrm{Aut}(T)$ acts continuously on $\Lambda : \omega_n \to \omega$ and $g_n \to I$ implies $g_n\omega_n \to \omega$, and Λ_T is compact iff each vertex has finitely many neighbours.

2. Add now the following assumptions:

(A) X has infinitely many ends,

(B) G is non compact,

(C) G is transitive on Ω_X.

Then:

PROPOSITION 2: (1) *The tree T is a biregular tree, with at most 2 finite valencies.*

(2) *There exists an equivariant homeomorphism $\lambda_1 : \Omega_T \to \Omega_X$ (where G acts on Ω_T via φ).*

Proof: STEP 1. We start by showing that $\Lambda_T \neq \phi$. If G would have a finite orbit of vertices, then by local finiteness of X, G would be a profinite group in a natural way, and so assumption B insures that every orbit is infinite. If T is finite, then $\mathrm{Aut}(T)$ is finite and so G has a subgroup H of finite index which leaves an edge of T invariant. Then H leaves a narrow cut D invariant, and therefore also its finite coboundary δD. Therefore assumption (B) insures that T is infinite and $\Lambda_T \neq \phi$, as in [5, proof of Theorem 4.1].

STEP 2. Construct a boundary map $\lambda_1 : \Lambda_T \to \Omega_X$ as follows: Fixing $s(D)$ as a reference vertex, each end of T has a unique representation as an antidirected path of edges $\{A_i\}_{i=0}^{\infty}$ starting at $s(D)$, i.e. $s(D) = t(A_0)$.

Define: $\lambda_1(\{A_i\}_{i=0}^{\infty}) = \cap_{i=0}^{\infty}\bar{A}_i$ where $\bar{A}_i$ means the closure in $X \cup \Omega_X$. $\cap_{i=0}^{\infty}\bar{A}_i \neq \phi$ by compactness, does not intersect X by Theorem 2, and λ_1 will be well defined if we show that the intersection contains a unique point of Ω_X. Given $\omega' \neq \omega \in \cap_{i=0}^{\infty}\bar{A}_i$ there exists a finite $F \subset X$ s.t. ω and ω' belong to the closure of different components of $X \setminus F$. Some A_{i_0} is disjoint from F by Theorem 2 and being connected A_{i_0} is contained in the component of $X \setminus F$ whose closure contains ω, and $\omega' \notin \bar{A}_{i_0}$. So λ_1 is well defined and clearly continuous: Let $L_n = \{v \in V | d(v, \delta D) \leq n\}$, and let $l_n = |\{C \in \mathcal{E} | \delta C \cap L_n \neq \phi\}|$, and consider two geodesics $\{A_i\}_{i=0}^{\infty}, \{B_i\}_{i=0}^{\infty}$ which have the first N edges in common. If $N > l_n$ then $\cap_{i=0}^{\infty}\bar{A}_i$ and $\cap_{i=0}^{\infty}\bar{B}_i$ are in the closure of the same component of $X \setminus L_n$. λ_1 is also injective, since given $\{A_i\}_{i=0}^{\infty}$, $\{B_i\}_{i=0}^{\infty}$ as above, consider the first vertex of T in which they diverge. Let $t(A_i) = t(B_i)$, $\quad 0 \leq i \leq N$, and $t(A_{N+1}) = s(A_N) \neq s(B_N) = t(B_{N+1})$. If $t(A_N) = t(B_N)$ but $s(A_N) \neq s(B_N)$, then by definition B_N is a subset of index one in A_N^* so $A_N \cap B_N = \phi$; distinct subsets of index one of a set $A_{N-1} \in \mathcal{E}$ are disjoint, and therefore λ_1 is injective.

Also, λ_1 is equivariant: $g\lambda_1(\{A_i\}_{i=0}^{\infty}) = \lambda_1(\varphi(g)\{A_i\}_{i=0}^{\infty})$, since $\varphi(g) \cdot \{A_i\}_{i=0}^{\infty}$ is the end represented by $\{gA_i\}_{i=0}^{\infty}$ and $\cap_{i=0}^{\infty} g\bar{A}_i = g \cap_{i=0}^{\infty} \bar{A}_i$.

STEP 3. G contains a compact subgroup having finitely many orbits on Ω_X. Let $K = G_v$ be the stabilizer in G of a vertex $v \in X$. K is a compact open subgroup with $\aleph_0$ cosets in G. If $\omega \in \Omega_X$ then $G\omega = \bigcup_{i=0}^{\infty} g_i K\omega = \Omega_X$ and each $g_i K\omega$ is compact, so by Baire's category theorem, at least one contains a non-empty open set, but then so does $K\omega$ and being homogeneous it is open, so each $g_i K\omega$ is open, so there exists a finite covering

$$\Omega_X = \bigcup_{i=1}^{N} g_i K\omega = \bigcup_{i=1}^{N} g_i K g_i^{-1} g_i\omega = \bigcup_{i=1}^{N} K^{g_i}\omega_i.$$

$Q = \cap_{i=1}^{N} K^{g_i} \cap K$ is compact and of finite index in K and K^{g_i}, and clearly has finitely many orbits on Ω_X.

Now $\varphi(K) \subset \mathrm{Aut}(T)$ is compact, and has finitely many orbits in Λ_T since λ_1 is an equivariant injection $\lambda_1 : \Lambda_T \hookrightarrow \Omega_X$. By construction T has at most two valencies, and since it admits a compact group of automorphisms with finitely many orbits, Λ_T is compact, the valencies are both finite, $\lambda_1(\Omega_T)$ is a G invariant set in Ω_X, so λ_1 is onto and therefore an equivariant homeomorphism, where $\Omega_T = \Lambda_T$ is the space of ends. □

Denote $\lambda_1^{-1} : \Omega_X \to \Omega_T$ by λ. We note that the continuity of φ implies that $\varphi(G)$ is a closed subgroup of $\mathrm{Aut}(T)$, since it is a countable union of the cosets of $\varphi(K)$ which is compact.

3. X has a biregular tree associated with it, and it is possible to describe the structure of X using that tree. We first collect some facts about biregular and boundary-transitive trees. Call a tree reduced if it has the property that geodesics can be continued indefinitely in both directions. We have:

PROPOSITION 3: *Let T be a boundary-transitive reduced tree, $G \subset \mathrm{Aut}\ (T)$ a closed non-compact boundary-transitive subgroup. Then:*

(a) *G contains a non-trivial translation along a geodesic, i.e. an automorphism g with $\min d_T(x, gx) > 0$.*

(b) *For every vertex $x \in T$ with valency at least 3, $G_x = \mathrm{St}_G(x)$ is transitive on the boundary.*

(c) *T is a subdivision of a biregular tree $T_{n,m}$ (i.e. T is obtained from $T_{n,m}$ by adding $\mathrm{Aut}(T_{n,m})$-orbits of vertices of valency 2).*

(d) *If $[x, y]$ is a segment connecting two vertices of valency ≥ 3 at minimal distance then $[x, y]$ intersect every G-orbit, and so there are at most two G-orbits of vertices of valency ≥ 3.*

(e) *For a vertex u of valency 2, G_u is boundary-transitive iff T is a subdivision of a regular tree, u bisects the segment $[x, y]$ above, and G is transitive on vertices with valency > 2.*

Proof: (a) As in the previous proposition, for all $x \in T$, $\overline{Gx} \cap \Omega_T = \Omega_T$, and therefore Gx intersects every connected component of $T \setminus F$, F finite. We now use an argument of Soardi and Woess [3]: let $e = \{x, y\}$ and delete the edge e from T, separating T into two connected components D_1 and D_2 both infinite, since T is reduced. Take $g \in G$ s.t. $ge \cap e = \phi$ and $ge \subset D_1$, and then $e \subset g\tilde{D}$ where $\tilde{D} = D_1$ or D_2. Now $g\tilde{D}^* \subset T \setminus e$, so $g\tilde{D}^* \subset D_1$ or D_2. If $\tilde{D}^* = D_1$, then $g(e \cup D_1) \subset D_1$ since $g(e \cup D_1)$ is a connected subset of $T \setminus e$ intersecting D_1. Otherwise $\tilde{D}^* = D_2$ and we can assume $gD_2 \subset D_1$. Take $h \in G$ with

$he \cap e = \phi$ and $he \subset D_2$. Again either $h(e \cup D_2) \subset D_2$, or $hD_1 \subset D_2$, and using $g(D_2 \cup e) \subset D_1$ we get $hg(D_2 \cup e) \subset hD_1 \subset D_2$. In any case we get a translation in G, since an automorphism g with $g(D \cup e) \subset D$ has an infinite orbit and therefore all its orbits are infinite so $\min d(x, gx) > 0$.

(b) Suppose $x_0 \in T$, valency $(x_0) \geq 3$, and let x_0 belong to the translation axis ℓ of $g \in G$, whose endpoints we denote by $\omega_+ = \lim_n g^n x_0$ and $\omega_- = \lim_n g^{-n} x_0$. G_{x_0} has $\aleph_0$ cosets in G, and finitely many orbits on Ω_T, as in the previous proposition. It follows that there exists a vertex $x \in \ell$, s.t. the endpoint of any geodesic starting at x_0 and passing through x can be mapped to ω_+ by an automorphism fixing x_0: WLOG $x = g^n x_0$ for some $n \geq 0$. It follows that it is possible to fix a segment on ℓ of the form $[g^k x_0, g^{k+n} x_0], k \in \mathbb{Z}$, and permute the neighbours of its right endpoint not on the segment. For some $n' \geq 0$, it is possible to fix a segment of the form $[g^{k-n'} x_0, g^k x_0], k \in \mathbb{Z}$, and permute the neighbours of its left endpoint not on the segment. Taking first $k = -n$ and then $k = n'$ and using the fact that the valency of x_0 is a least 3, we see that these sets of neighbours intersect. Consequently, G_{x_0} is transitive on the neighbours of x_0.

Now let $\omega \neq \omega_+$ be an end. We can assume the geodesic from x_0 to ω is disjoint from $[x_{-1}, \omega_-)$. $g^n \omega$ belongs to the set of geodesics with initial segment $[x_0, g^n x_0]$, and so for some $k \in G_{x_0}, kg^n\omega = \omega_+$ by transitivity of G_{x_0} on this set. k necessarily fixes $g^n x_0$ and so for $h = g^{-n} k g^n$ we have: $hx_0 = x_0$ and $h\omega = \omega_+$ and the proposition follows.

(c + d) Let $x \neq y$ be two vertices on ℓ of valencies ≥ 3 at minimal distance. The valency of vertices is constant on circles around x, and on circles around y, by the boundary transitivity of the stability groups. It follows that circles of radius $2kd_T(x, y)$ around x consist of vertices of valency equal to that of x, those of radius $(2k + 1)d_T(x, y)$ around x consist of vertices of valency equal to that of y, and the rest of the circles around x consist of vertices of valency 2. It follows also that the segment $[x, y]$ intersects all the G-orbits in T, since any point can be brought to the segment by repeated use of rotations around vertices of valency equal to that of x or y. Consequently T is a subdivision of a biregular tree, and G has at most 2 orbits of vertices whose valency is greater than 2. It follows that the stability group of any vertex of valency at least 3 is boundary-transitive.

(e) Clearly if valency $(u) = 2$ and G_u is boundary-transitive, then T is a subdivision of a regular tree, u bisects the segment $[x, y]$ connecting its two closest neighbours of valency > 2 and G is transitive on vertices of valency > 2. Conversely, G_u is boundary-transitive in that situation: Consider a segment

$[x, y] \ni u$, and $g \in G$ with $gx = y$. If g has a fixed point at equal distances from x and y then u is also a fixed point and we are done. g cannot have a fixed point z closer to x than to y or vice versa, so the only other possibility is that g is translation along a geodesic ℓ containing $[x, y]$. Rotating x to x', around y, where $x' \in \ell$ and $d(x, y) = d(y, x')$ and composing with g^{-1}, we get a map fixing u and interchanging x, y. So G_u is boundary-transitive.

Remarks: (1) In [3] there is a construction of a graph with infinitely many ends and a non-compact vertex transitive group of automorphisms with exactly two orbits on the boundary, one of which is a single point. It is also easy to construct a tree with a non-compact group of automorphisms with two orbits on Ω_T one of which is a two point set: Let T_0 be a rooted tree with $\mathrm{Aut}(T_0)$ transitive on Ω_{T_0}, and from each point $n \in \mathbb{Z}$ construct a copy T_0^n of T_0 with n as a root. These examples show that it is impossible to weaken the assumption of boundary-transitivity to the existence of finitely many orbits on the boundary, not even for trees, and not even assuming vertex transitivity of the graph.

(2) The assumption that $G = \mathrm{Aut}(X)$ is non-compact and has finitely many closed orbits in Ω_X (or equivalently that all the G-orbits in Ω_X are open) is easily seen to yield the conclusion that X is boundary transitive, as follows: Let $\widetilde{Gx}$ be the set of limit points of Gx in Ω. If $\widetilde{Gx} \neq \Omega$ it is a proper open invariant set as well as its complement $\Omega \setminus \widetilde{Gx}$. $\widetilde{Gx}$ and $\widetilde{Gy}$ are either identical or disjoint, so take point $y \in X$ so that $\widetilde{Gy} \subset \Omega \setminus \widetilde{Gx}$. Let $d = d(Gx, Gy)$ and $D = \{z \in Gy | d(z, Gx) = d\}$. Then D is G - invariant and clearly finite, since any limit point of D in Ω will belong to $\widetilde{Gx} \cap \widetilde{Gy}$. Since G is non-compact, we must have $\widetilde{Gx} = \Omega$, so if $\omega \in \widetilde{Gx}$ then $G\omega$ intersects all the orbits of G in Ω — all orbits are open — and therefore $G\omega = \Omega$.

(3) If for a tree $T, G = \mathrm{Aut}(T)$ is non-compact and has finitely many orbits of vertices, then T is boundary transitive iff T/G is a (degenerate or non-degenerate) segment or a circle, in which all points correspond to vertices of valency 2, except one point on the circle or the two end points of the segment. The 'if' part follows from the fact that a maximal tree in T/G has a lift which is a fundamental domain of G in T and all the points in the lifted segment except the two endpoints have valency 2. The 'only if' part follows from Proposition 3.

(4) Assuming T/G is finite, it follows that if G has more than one orbit in Ω, it has $2^{\aleph_0}$. There are two cases to consider: If T/G has a vertex of valency ≥ 3, then decomposing the vertices of T to types according to the orbit to which they belong, there exists a vertex v of type 0, with three neighbours having different types 1,2,3. If a geodesic intersects Gv in an infinite set then at each point in

the intersection a choice is made as to the type of the next vertex. Consider $C = \{1,2,3\}^{\mathbb{N}}$, let S be the shift: $(S\alpha)_i = (\alpha_{i+1})$ for $\alpha = (\alpha_i) \in C$, and call α, β shift-equivalent iff $S^n\alpha = S^m\beta$ for some n and m. Given such a geodesic ℓ, define a sequence α_ℓ as the sequence of types of those vertices in $\{1,2,3\}$ which follow a vertex of type 0 on ℓ. Call such a sequence admissible. Clearly there are $2^{\aleph_0}$ admissible sequences, since $\aleph_0$ choices are made. For each admissible α, define $\Omega_\alpha = \{\omega |$ the sequence α_ℓ obtained from a geodesic ℓ converging to ω is shift equivalent to $\alpha\}$. Ω_α is G invariant, and disjoint from Ω_β if α and β are not shift equivalent.

In the other case T/G is a segment or a circle and we can assume WLOG that all points correspond to vertices of valency ≥ 3, and that there are at least three points in T/G. One of them will be connected to two of the others, call it type 2, and it is easily seen that as we go along a geodesic (intersecting the orbit of type 2 in an infinite set) there are $\aleph_0$ choices to be made about whether the next vertex will be of type 1 or type 3. As in the previous case it follows that there are $2^{\aleph_0}$ G-orbits on the boundary.

4. We now describe the structure of X using the associated biregular tree T. Recall that $\varphi(G)$ is transitive on Ω_T, and that for any $s \in T$, $\varphi(G)_s$ is transitive on the neighbours of s in T, and denote $K_s = \varphi^{-1}(\varphi(G)_s)$. Define for each $D \in \mathcal{E}$ a set of vertices in V, $v(D) = D \setminus \cup_{A \prec D} A =$ the set of vertices in D not contained in any subset of D of index one. We then have:

Proposition 4: (1) *$v(D)$ is finite.*
(2) *$v(D) = \cap\{A | s(D) = s(A)\}$ and consequently $v(D)$ depends on $s(D)$ only.*
(3) *If $s(D_1)$ and $s(D_2)$ are two distinct vertices of T, then $v(D_1)$ and $v(D_2)$ are disjoint.*
(4) *$K_{s(D)}$ leaves $v(D)$ invariant.*

Proof: (1) Consider $\bar{D} \cap \Omega_X$. Looking at T, and using the homeomorphism λ, we see that $\bar{D} \cap \Omega_X = \cup_{A \prec D} \bar{A} \cap \Omega_X$. Therefore $\bar{D} \setminus \cup_{A \prec D} \bar{A}$ is closed and has empty intersection with Ω_X, so it is finite.

(2) Let valency $s(D) = n \geq 2$ and then we have $D = v(D) \cup \cup_{i=1}^{n-1} A_i$ (see Fig. 1), and this union is disjoint, since distinct subsets of index one in D are disjoint (as was seen in step 2 in the proof of Proposition 2). Now decompose $A_1^* = v(A_1^*) \cup D^* \cup \cup_{j=2}^{n-1} A_i$ which is again a disjoint decomposition. A_1 is a subset of D disjoint from $v(D)$, so $v(D) \subset A_1^*$. By definition $v(D) \subseteq v(A_1^*)$ and reversing the argument gives an equality, for any A with $s(A) = s(D)$, and so $v(D) = \cap\{A | s(A) = s(D)\}$.

(3) If $s(D_1) \neq s(D_2)$ in T, and there is a directed path in T containing both edges, D_2 following D_1, then by definition D_1 is a subset of an index one subset of D_2, so $D_1 \cap v(D_2) = \phi$ and in particular $v(D_1) \cap v(D_2) = \phi$. Otherwise D_1 and D_2 have incompatible orientations — there is no directed path containing both and then by definition $D_1 \subset D_2^*$ so again $D_1 \cap v(D_2) = \phi$ and $v(D_1) \cap v(D_2) = \phi$.

(4) The group $\varphi(G)_{s(D)}$ permutes the edges going out of $s(D)$ and therefore $K_{s(D)}$ permutes the sets occurring in the decomposition of part (1) leaving $v(D)$ invariant since $A \prec B \Leftrightarrow kA \prec kB$, so the image of a point not belonging to any subset of index one has the same property.

PROPOSITION 5: *(1) For $D \in \mathcal{E}$, at least one of the sets $v(D), v(D^*)$ is non-empty.*
(2) The sets $v(D)$, $D \in \mathcal{E}$, constitute a partition of the set V of vertices of X to finite sets.
(3) The partition is G-invariant, and the family $\{v(D) | D \in \mathcal{E}\}$ consists of at most two G - orbits.

Proof: (1) If $v(D) = v(D^*) = \phi$ then $D = \cup_{A \prec D} A$, $D^* = \cup_{B \prec D^*} B$ (see Fig. 1). Any vertex $v \in D$ will satisfy $v \in A$ for some $A \prec D$. But A is a G translate of either D or D^*, so A is also the union of its index one subsets, so $v \in A' \prec A \prec D$, etc. It follows that v belongs to the intersection of a decreasing sequence of narrow cuts, contrary to Theorem 2.

(2) We saw above that for $D_1 \neq D_2$ the sets $v(D_1)$ and $v(D_2)$ are disjoint. The collection $\{v(D) | D \in \mathcal{E}\}$ covers V, since given $v \in V$, $v \in D$ or $v \in D^*$ and so belongs either to a subset of index one or to $v(D)$ or $v(D^*)$. If v belongs to a subset of index one — decompose this subset to its subsets of index one, etc. and by the same argument as in (1) — after finitely many steps a set A is found s.t. $v \in v(A)$.

(3) The partition induces the equivalence relation $v \sim w \Leftrightarrow \exists A$ s.t. $v, w \in v(A)$, and since $v(gA) = gv(A) (A \prec B \Leftrightarrow gA \prec gB)$ it follows $v \sim w \Leftrightarrow gv \sim gw$, so $\sim$ is a G-invariant equivalence relation. □

We note that the set $\{v(C) | C \in \mathcal{E}\}$ is in 1-1 correspondence with the vertices of T, provided $v(D) \neq \phi \neq v(D^*)$, given by $\tau : v(A) \mapsto s(A)$ (which is well defined and injective by Proposition 4). If only one of the sets is non-empty, say $v(D) \neq \phi$, then the same map will have as its image the set of vertices in T at even distance from $s(D)$. The map τ then describes $V(X)$ as a cover of $V(T)$, s.t. the fiber over each vertex $s \in T$ is finite, and since τ is obviously G

equivariant and G has at most two orbits of vertices in T by Proposition 3 (d), each fiber is mapped (by some element of G) isomorphically onto either $v(D) =$ the fiber over $s(D)$ or $v(D^*) =$ the fiber over $s(D^*)$, one of which may be empty. In general, τ does not map neighbours in X to neighbours in T, but we can use τ to define the depth of edges $e = (v, w)$ in X, by $d(e) = d_T(\tau(v), \tau(w)) =$ the distance in T between $\tau(v)$ and $\tau(w)$. Clearly $d(e) = 0$ iff e connects two vertices in the same set $v(D)$ for some D, and $d(e) = k + 1$ iff e has one vertex in a set D and the other in a subset of index k in D^* (where we define for consistency that $v(D) \subset D$ is the subset of D of index 0). Obviously, d is a G - invariant function.

5. To construct the tree T, it was necessary to choose a narrow cut. We now show:

PROPOSITION 6: *The tree T is determined by X and does not depend on the choice of the narrow cut used to define it.*

For the proof we need the following:

PROPOSITION 7: *Let $H \subset \mathrm{Aut}(T)$ be a closed non-compact boundary transitive subgroup, where T is a biregular tree. Let $e = \{s, t\}$ be an edge of T. Then*
(1) The group H_e is transitive on the set of neighbours of s other than t, and therefore H_s is doubly transitive on the set of neighbours of s.
(2) H_e is a maximal subgroup of H_s and any two such are conjugate. Consequently the valency of s is the index of a maximal subgroup of the stabilizer of s in H.

Proof: (1) We can assume that the valency of s is at least 3, otherwise there is nothing to prove. Consider a geodesic containing e, as in Fig 2. H_e has open orbits on the boundary and consequently for some n, H_e is transitive on the set of neighbours of s_n other than s_{n-1}, and we can assume s_0 and s_n to be in the same H orbit (H has at most two orbits by Proposition 3) and then H_{s_n} is conjugate to H_{s_0}. Now $H_{[s_0 s_n]}$ is transitive on the outer neighbours of s_n, and so by transitivity of H_{s_0} on the circle of radius n around s_0, the stabilizer of a radius of length n is transitive on the outer neighbours of its endpoint, and this is true for the conjugate group H_{s_n} too. So $H_{[s_n s_0]} \subset H_e$ is transitive on the neighbours of s_0 except s_1, and in particular H_s is doubly transitive.

(2) If a group G is doubly transitive on a set $E, |E| \geq 3$ then there is no domain of imprimitivity, i.e. a proper subset Y with at least two elements

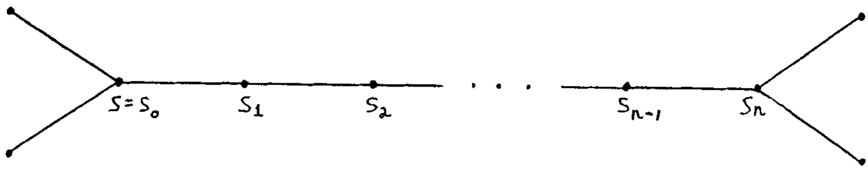

Fig. 2.

satisfying $gY \cap Y = \phi$ or $gY = Y$ for all $g \in G$, since if $a \neq b$ are in Y and $c \in E \setminus Y$ then $g(a,b) = (a,c)$ for some g so $a \in gY \cap Y \neq \phi$, but also $gY \neq Y$ since $c \in gY \setminus Y$. If there are no domains of imprimitivity then stabilizers of elements in E are maximal subgroups: If H satisfies $G_e \subsetneqq H \subsetneqq G$ then $H \cdot e = Y$ has $|Y| \geq 2$ and $Y \subsetneqq E$, otherwise each $g \in G$ is congruent to some $h \in H$ modulo $G_e \subset H$ so $G = H$, and finally $gY \cap Y \neq \phi$ implies $gh_1e = h_2e$ for some h_1, $h_2 \in H$ so $h_2^{-1}gh_1 \in G_e \subset H$ and $g \in H$ in which case $gY = Y$. □

Proof of Proposition 6: Let D' be another narrow cut, and associate with (X, D') the tree T', the representation $\varphi' : G \to \mathrm{Aut}(T')$ and the boundary map $\lambda' : \Omega_{T'} \to \Omega_X$. If N is the kernel of the action of G on Ω_X, then $\varphi(G) \cong G/N \cong \varphi'(G)$. This isomorphism follows immediately from the fact that the action of $\mathrm{Aut}(T)$ on Ω_T is faithful: no non trivial element acts as the identity, if $|\Omega_T| > 2$. $\bar{G} = G/N$ acts on T and T' via φ and φ', and the composition $\lambda \circ \lambda' : \Omega_{T'} \to \Omega_T$ gives a $\bar{G}$ equivariant homeomorphism on the boundary. Let $s' \in T'$ be a vertex of valency ≥ 3, and consider $\varphi'(G)_{s'}$, which is a compact subgroup transitive on $\Omega_{T'}$ (by Proposition 3) having no inversions and consequently in its action on T it stabilizes a vertex s, and has to coincide with its stabilizer in $\varphi(G)$ since any proper subgroup of $\varphi(G)_s$ stabilizes an edge, by Proposition 7, and cannot be transitive on the boundary. So the valencies of s and s' are both equal to the index of a maximal subgroup of $\varphi(G)_s \cong \varphi'(G)_{s'}$. By the same argument, if there exists a vertex with another valency in one of the trees, then there is one in the other tree too, and the valencies are equal. □

6. Since the associated tree does not depend on the choice of the narrow cut we can define it to be the tree type of X, denoted $\mathrm{Tree}(X)$. It is natural to inquire how far is X from its tree type T_X and how far is G from a subgroup of $\mathrm{Aut}(T)$.

PROPOSITION 8: *(1) $\operatorname{Ker}\varphi =$ the kernel of the action of G on Ω_T is a compact group which is locally finite with finite exponent.*
(2) The number of orbits of G in X is equal to the number of orbits of $G_{v(D)}$ in $v(D)$ plus the number of orbits of $G_{v(D^*)}$ in $v(D^*)$, if there is no inversion satisfying $\sigma(D) = D^*$, and otherwise the number of orbits of $G_{v(D)}$ in $v(D)$.

Proof: (1) $\operatorname{Aut}(T)$ is faithful on Ω_T, and therefore $\operatorname{Ker}\varphi$ is the kernel of the action of G on $\Omega_X \cong \Omega_T$. $\operatorname{Ker}\varphi$ acts trivially on T, and by equivariance of $\tau : V(X) \to V(T)$ stabilizes each $v(C)$, $C \in \mathcal{E}$, and is therefore compact, since X is locally finite. If we denote by $H(C)$ the group of permutations induced by $N = \operatorname{Ker}\varphi$ on $v(C)$ then the map $\psi : N \to \Pi_{C\in\mathcal{E}} H(C)$ by $n \mapsto \{n|_{v(C)} | C \in \mathcal{E}\}$ is a monomorphism, since $\{v(C)|C \in \mathcal{E}\}$ is a partition of V. Each $H(C)$ is isomorphic either to $H(D)$ or to $H(D^*)$ and so $N \subset \Pi_{C\in\mathcal{E}} H(C)$ is locally finite of finite exponent.

(2) Let $G_{v(D)}$ have a orbits in $v(D)$, $G_{v(D^*)}$ have b orbits in $v(D^*)$. It is then possible to color the vertices of X in $a+b$ colors G-invariantly, by defining first on $v(D) : c(v) = c(w) \Leftrightarrow v \in G_{v(D^*)} \cdot w$ and on $v(D^*) : c(v) = c(w) \Leftrightarrow v \in G_{v(D^*)} \cdot w$ and then: $c(gv) \stackrel{d}{=} c(v)$, for $v \in v(D) \cup v(D^*)$, $g \in G$. The function c is well defined, provided there are no inversions satisfying $\sigma(D) = D^*$ since if $v \in v(D) \cup v(D^*)$, $g \in G$ then $hv = gv \Leftrightarrow h \in gG_v$ but $G_v \subset \varphi^{-1}(\varphi(G)_{\tau(v)}) = G_{v(D)}$ or $G_{v(D^*)}$. By definition, c is G-invariant. The argument for the case an inversion exists is analogous.

Note that for each color $1 \leq i \leq a + b$ the set of vertices $V_i = \{v | c(v) = i\}$ is a G orbit, and τ restricts to an equivariant map $\tau_i : V_i \to T$ whose image is a $\varphi(G)$-orbit of vertices in T, and τ_i also satisfies that the stabilizer in G of the vertex $\tau_i(v)$ is transitive on the fiber $\tau_i^{-1}(\tau_i(v))$.

7. Some Remarks and Corollaries

(a) The maps τ and λ are compatible in the following sense:

PROPOSITION 9: *If $x_n \in X, x_n \to \omega \in \Omega_X$ then $\tau(x_n) \to \lambda(\omega)$.*

Proof: We note that by construction of T, given a narrow cut A with $v(A) \neq \phi$, there is an edge of T associated with A, and $\tau(A)$ is the set of vertices of $\varphi(G) \cdot s(A)$, which belong to an anti-directed chain whose first edge is A (and first vertex $t(A)$). This follows since $A = \cup\{v(D) | D \in \mathcal{E}, D \subset A\}$. If $\{A_i\}_{i=1}^{\infty}$ is the sequence of edges on an anti-directed geodesic converging to $\lambda(\omega)$, then $\{\bar{A}_i\}_{i=1}^{\infty}$ is a decreasing sequence of neighbourhoods of ω, and $\cap_{i=1}^{\infty} \bar{A}_i = \{\omega\}$.

$\tau(\overline{A_i})$ is a decreasing sequence of neighbourhoods of $\lambda(\omega)$ by the above and the proposition follows. □

(b) Given two finite graphs X_1 and X_2 that are vertex-transitive (i.e. $\mathrm{Aut}(X_i)$ is transitive on $V(X_i)$), their free product $X = X_1 * X_2$ is the graph obtained by connecting countably many copies of X_1 and X_2, to form a connected graph, in which every vertex is the intersection of exactly one copy of X_1 and one copy of X_2, and the only simple circuits in X are those contained entirely in some copy of X_1 or X_2. For example, the Cayley graph of $Z_n * Z_m$ $(Z_n = Z/nZ)$ w.r.t. the generators $(Z_n \cup Z_m) \setminus \{0\}$ is the free product of two complete graphs on n and m vertices (without loops). This graph is easily seen to be boundary-transitive, which also follows from the fact that it is the dual graph (see the next section) of a biregular tree $T_{n,m}$. In general the automorphism group of such free products has 'large' (= uncountable) compact subgroups and it is natural to inquire whether they provide examples of boundary-transitive graphs. We have:

PROPOSITION 10: *$X_1 * X_2$ is boundary-transitive iff X_1 and X_2 are complete graphs on $|V(X_1)|$ and $|V(X_2)|$ vertices.*

Proof: Suppose $X = X_1 * X_2$ is boundary-transitive. X is clearly vertex transitive, and the set of edges incident to a vertex falls naturally into two sets according to the type of subgraph to which they belong. To construct the Dunwoody tree, we need to choose a $G = \mathrm{Aut}(X)$ orbit of narrow cuts. If the valency of a vertex in X_2 is smaller than the valency of a vertex in X_1 we have to choose D, D^* as shown in Fig. 3, and if they are equal, there is another choice $D_1 = D \setminus v, D_1^*$.

One of the choices D or D_1 will satisfy the conclusion of Dunwoody's Theorem 1, and we label this choice by D, and we can assume that ∂D consists of one vertex-labeled v, by going over to D^* if necessary. Since X_i is vertex-transitive, G_{X_i} is transitive on the vertices of X_i, and therefore the subgraph X_1 is the fiber under τ of a single vertex of T — the Dunwoody tree. The image under τ of $V(X)$ is the $\mathrm{Aut}(T)$ - orbit of this vertex, since the decomposition of $V(X)$ to subgraphs of type X_1 is a G - invariant partition. By Proposition 7, $\varphi(G)_{\tau(X_1)}$ is doubly transitive on the tree neighbours, and it follows immediately that G_{X_1} and therefore $\mathrm{Aut}(X_1)$ is doubly transitive on $V(X_1)$, which implies that $V(X_1)$ is a complete graph. Now the stabilizer in $\varphi(G)$ of a vertex of T in the other $\varphi(G)$-orbit of vertices acts doubly transitively on the tree neighbours, so it acts doubly transitively on D and the index one subsets of D^*. This clearly implies that $\mathrm{Aut}(X_2)$ is doubly transitive on X_2 so it too is a complete graph. □

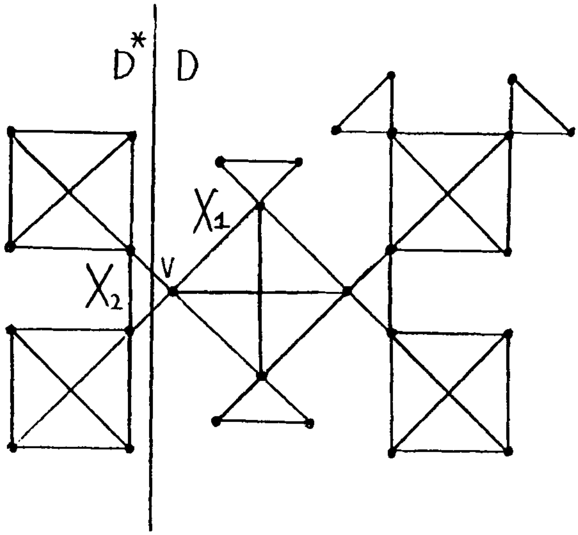

Fig. 3.

Remark: Suppose X_i, $1 \leq i \leq k$ are vertex transitive graphs, $|X_i| \geq 2$.

An argument similar to the one in Remark 4 in section 3 above shows that if $X = X_1 * X_2 * \cdots * X_k$, $k \geq 2$ is boundary-transitive, then $|X_i|$ can take either two values, in which case $k = 2$ and X is a free product of two complete graphs, or $|X_i| = n$ is constant, and then an argument similar to the one in the Proposition 9 shows that each X_i is a complete graph on n vertices.

8. Construction of Graphs

Let B denote the class of locally finite graphs X with infinitely many ends and a non compact group of automorphisms G transitive on the boundary. B is stable under the following operations:

(1) Add a G orbit of vertices of valency 2, i.e., choose a G-orbit of edges and add a vertex "in the middle" of each edge.

(2) For each G - orbit of vertices choose a finite graph with a preferred vertex and append a copy of it to the vertices in the chosen orbit.

More formally, if $\theta_1, \ldots, \theta_r$ are the G orbits in X and $F_1, \ldots, F_r$ are finite graphs with preferred vertices $a_1, \ldots, a_r$ define

$$V(X') = \bigcup_{i=1}^{r} \theta_i \times F_i$$

and $((u,a),(v,b)) \in E(X')$ iff $(u,v) \in E(X)$ and $a,b \in \{a_1,\dots,a_r\}$ or $u = v$ (say $u \in \theta_i$) and $(a,b) \in E(F_i)$. X' is obviously connected, and G acts on X' by $g(u,a) \stackrel{d}{=} (gu,a)$.

Clearly, if $H_i = St_{\mathrm{Aut}(F_i)}(a_i)$ then the group $H_i^{\theta_i} = \Pi_{v\in\theta_i} H_i$ injects into $\mathrm{Aut}(X')$.

(3) Multiply by a finite graph F (which may have loops), i.e. take

$$V(X') = V(X) \times V(F),$$

$((u,a),(v,b)) \in E(X') \Leftrightarrow (u,v) \in E(X)$ and $(a,b) \in E(F)$, with G action $g(u,a) = (gu,a)$,

(4) If $(v,w) \in V \times V$ add the G orbit $\{(gv,gw) | g \in G\}$ to the set of edges of X. In particular the set $U_\ell = \{(v,w) | d_X(v,w) = \ell\}$ is a finite union of G - orbits of pairs, and so for any finite sequence of lengths $\ell_i \geq 2$, $1 \leq i \leq r$, it is possible to connect by new edges the vertices at distance ℓ_i apart. The G action extends in a natural way.

(5) Delete a G-orbit of edges, provided the remaining graph is still connected. As an example, connect in T_3 vertices at distance 2 and 3 and delete the original edges.

(6) Join two graphs with isomorphic associated trees. If $\mathrm{Tree}(X_i) \cong T$ take

$$H \subset \varphi_1(G_1) \cap \varphi_2(G_2)$$

where $\varphi_i : G_i \to \mathrm{Aut}(T)$ are the representations constructed above which we assume here to be faithful, and H a closed non-compact subgroup of $\mathrm{Aut}(T)$ which is transitive on Ω_T; then take

$$V(X') = V_1 \cup V_2, E(X') = E(X_1) \cup E(X_2) \cup E$$

where E is a set of edges, which consists of the H orbit of a new edge connecting a vertex in V_1 to a vertex in V_2 (in order to make X' connected) and where H acts on $V_1 \cup V_2$ by $h \cdot v_i = \varphi_i^{-1}(h)v_i$.

(7) Pass to the quotient graph X/N obtained as the space of orbits of the action of a compact normal group $N \subset G$, if such exists.

(8) Pass from X to X^* = the dual of X, which has $E(X)$ as a set of vertices, and for $(e,f) \in E(X) \times E(X)$, (e,f) constitutes an edge in $E(X^*)$ iff e,f have a common vertex in X. X^* is a connected graph if X is, $\mathrm{Aut}(X)$ acts on X^*, and there is a natural equivariant identification of Ω_X and Ω_{X^*} which comes from the fact that the sets of components $\{C_F| \ |F| < \infty,\ F \subset X\}$ and $\{C_F| \ |F| < \infty,\ F \subset X^*\}$ are mutually cofinal in one another.

Example: $T^*_{n,m} = Z_n * Z_m \stackrel{d}{=}$ the free product of two complete graphs on n and m vertices (with no loops).

(9) Let X be a compact two-dimensional simplicial complex, and X_1 the 1-skeleton of X. Let $\tilde{X}$ be the universal cover of X, and $\tilde{X}_1$ the 1-skeleton of the two-dimensional simplicial complex $\tilde{X}$. $\tilde{X}_1$ covers X_1 under the covering map $p : \tilde{X} \to X$. Suppose $\Gamma = \Pi_1(X, x_0) \cong F_r$ = The free group on r generators. Γ acts naturally on $\tilde{X}_1$, and fixing $\tilde{x}_0$ with $p(\tilde{x}_0) = x_0$, construct a Dirichlet fundamental domain w.r.t. $\tilde{x}_0$ using the edge path metric $d_{\tilde{X}_1} = d$:

$$D(\tilde{x}_0) = \{\tilde{x} | d(\tilde{x}_0, \tilde{x}) \le d(\tilde{x}_0, \gamma\tilde{x}), \forall \gamma \in \Gamma\} \quad \text{and } \bar{D}(\tilde{x}_0) = \{\tilde{x} | d(\tilde{x}, D(\tilde{x}_0)) \le 1\}.$$

$\bar{D}(\tilde{x}_0)$ contains the $2r$ points closest to $\tilde{x}_0$ in its Γ - orbit, $\tilde{x}_i = \gamma_i \tilde{x}_0$ and $\{\gamma_i\}_{i=1}^{2r} = S$ is a symmetric set of generators for Γ. Figure 4 shows that $\tilde{X}_1$ is sometimes boundary-transitive.

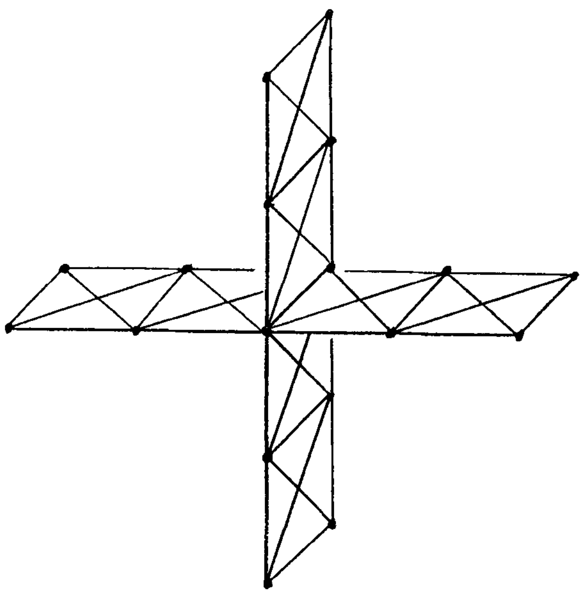

Fig. 4. $\tilde{X}$ = the universal cover of $[0,1]\times$ figure 8.

Remarks: (1) It is natural to expect that $\tilde{X}_1$ is boundary-transitive iff the group $\mathrm{Aut}\bar{D}(\tilde{x}_0)_{\tilde{x}_0}$ (graph automorphisms of $\bar{D}(\tilde{x}_0)$ fixing $\tilde{x}_0$) is doubly transitive on the set S.

(2) In all the examples above, the representation $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(T)$ has full image; i.e. $[\mathrm{Aut}(T) : \mathrm{im}\varphi] \le 2$. It is easy to see that each of the constructions (1)

– (8) (except 6) described above yields a graph X' with imAut$(X') \supset$ imAut(X). It is unknown, however, whether the index of imφ will be ≤ 2 for all boundary transitive graphs, or whether it is possible to obtain non-open subgroups of Aut(T) (such as $PSL_2(\mathbb{Q}_p)$, for example) as the image of the full automorphism group of a boundary-transitive graph.

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The author has been informed by the referee that similar results have been independently obtained by R. G. Möller, and will be published in a forthcoming issue of the Mathematical Proceedings of the Cambridge Philosophical Society.

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